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Entanglement Properties of Output Field from Beam Splitter with the Generalized Two-Mode Squeezed Vacuum State Inputs

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Abstract Entangling properties of output field from beam splitter are investigated with the generalized two-mode squeezed vacuum state inputs. It is found that the entanglement are not only strongly dependent on the squeezing parameters of the input state but also quite involved with the angular parameter of the beam splitter. By appropriately choosing the parameters of both input field and the beam splitter, the best entanglement can be obtained.

Keywords Beam splitter operator · Characteristic function · Entanglement · Generalized two-mode squeezed vacuum state

1 Introduction

As is well known, entanglement is at the heart of the current development of quantum information processing [1]. Entanglement-assisted communication can enlarge the channel capacity [2] and enhance channel efficiency [3]. Entanglement may play a key role in secure communication [4]. Especially, continuous variable entangled states have been viewed as an alternative resource for quantum information processing. Because these states are more easily produced than the discrete variable ones from reliable sources [5, 6]. The two-mode squeezed vacuum state (SVS) is just a practically existing familiar continuous variable entangled state, which can be generated either by mixing two single-mode SVS in a balanced beam splitter (BS) [7, 8] or a non-degenerate parametric optical amplifier from the vacuum [9].

Alternatively, a beam splitter (BS), one of the few experimentally accessible optical devices, can also be used to generate quantum entanglement between two modes [10]. There have been many previous studies [11–21] on the properties of the entanglement using BS as an entangler for various input states, especially continuous variable. For instance, Paris [7] studied entanglement properties of the output state from a Mach-Zehnder interferometer

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for squeezed input states. The output entanglement quantity was studied by Kim's group in Ref. [13] for a given SVS and an explicit formula expressing the output state in the form of the two-mode SVS had been obtained. Based on the Kim's work, the entanglement properties are investigated by Zhou's group by virtue of the linear entropy, and nonclassicalities are also examined of output fields from a BS for pure binomial state inputs [17]. From these works, one can find that in order to obtain an entangled two-mode state out of a BS, it is necessary to have a nonclassical state at one input [18, 19]. No entanglement is produced for coherent states due to their classicality.

Enlighten by the above significant works, in this paper, we investigate the properties of entanglement of output field from BS for the generalized two-mode squeezed vacuum state (GTSVS) inputs. Here GTSVS can be obtained by operating a kind of two-mode squeezing operator $U_2(r,s)$ with three parameters (see below (1)) on the two-mode SVS, namely, $U_2(r,s)|00\rangle$. Motivations to pay attention to the case lie on the following three facts: (i) $U_2(r,s)$ has its coherent state projection operator realization and makes up a loyal representation of the symplectic group. (ii) $U_2(r,s)$ is a generalized squeezing operator, in a particular case it reduces to the usual squeezing operator $\exp[\lambda(a^\dagger b^\dagger - ab)]$. (iii) $U_2(r,s)|00\rangle$ is a kind continuous variable entangled state as well. In the following section, the properties of $U_2(r,s)$ are briefly reviewed. Then, we devote to studying the entangling properties of output field from BS with the GTSVS inputs by introducing linear entropy as a measure of the entanglement with the help of the characteristic function. Finally, the numerical results for the degree of entanglement are shown to observe the influence of entanglement entropy governed by its parameters.

2 Properties of $U_2(r,s)$

We begin with briefly reviewing the properties of the generalized two-mode squeezing operator $U_2(r,s)$ with three parameters. As is described in Refs. [22, 23], $U_2(r,s)$ can be expressed as a ket-bra integration operator in the two-mode coherent state basis,

$$U_2(r,s) = s\int \frac{d^2z_1 d^2z_2}{\pi^2}\left|sz_1 + rz_2^*, rz_1^* + sz_2\right\rangle\langle z_1, z_2|, \tag{1}$$

where $|z_1\rangle = \exp(-|z_1|^2/2 + z_1 a^\dagger)|0\rangle_1$ and $|z_2\rangle = \exp(-|z_2|^2/2 + z_2 b^\dagger)|0\rangle_2$, r and s are complex numbers satisfying the unimodularity condition $ss^* - rr^* = 1$, $(z_1, z_2) \longrightarrow (sz_1 + rz_2^*, rz_1^* + sz_2)$ is the mapping of symplectic transform in (z_1, z_2) phase space. Using the normal ordering form of two-mode vacuum projector $|00\rangle\langle 00| =: \exp(-a^\dagger a - b^\dagger b):$ and the IWOP technique [24–26] we can directly calculate

$$\begin{aligned} U_2(r,s) &= s\int \frac{d^2z_1 d^2z_2}{\pi^2} : \exp\Big[-|s|^2\left(|z_1|^2 + |z_2|^2\right) - r^* s z_1 z_2 - r s^* z_1^* z_2^* \\ &\quad + \left(sz_1 + rz_2^*\right)a^\dagger + \left(rz_1^* + sz_2\right)b^\dagger + z_1^* a + z_2^* a - a^\dagger a - b^\dagger b\Big]: \\ &= \exp\left(\frac{r}{s^*}a^\dagger b^\dagger\right)\exp\left[\left(a^\dagger a + b^\dagger b + 1\right)\ln\left(\frac{1}{s^*}\right)\right]\exp\left(-\frac{r^*}{s^*}ab\right), \end{aligned} \tag{2}$$

where $:\ :$ stands for the normal ordering. Based on (2), the following unitary transform can be induced

$$U_2^\dagger(r,s)\begin{pmatrix} a \\ b^\dagger \end{pmatrix} U_2(r,s) = \begin{pmatrix} s^* & r \\ r^* & s \end{pmatrix}\begin{pmatrix} a \\ b^\dagger \end{pmatrix}, \tag{3}$$

so $U_2(r,s)$ is unitary and (2) is its coherent state representation. It is readily seen that by setting $s=\cosh\lambda$ and $r=\sinh\lambda$, equation (2) reduces to $\exp[\lambda(a^\dagger b^\dagger - ab)]$, which is just usual two-mode squeezing operator. Moreover, $U_2(r,s)$ has an important feature: the set of coherent state projection operator $U_2(r,s)$ can make up a loyal representation of the symplectic group, i.e., [27]

$$\begin{aligned} U_2(r,s)\,U_2\left(r',s'\right) &= ss'\int\frac{d^2z_1d^2z_2}{\pi^2}\int\frac{d^2z_1'd^2z_2'}{\pi^2}\left|sz_1+rz_2^*, rz_1^*+sz_2\right\rangle \\ &\quad\times\left\langle z_1,z_2\right|\left. s'z_1'+r'z_2'^*, r'z_1'^*+s'z_2'\right\rangle\left\langle z_1',z_2'\right| \\ &= \exp\left(\frac{r''}{s''^*}a^\dagger b^\dagger\right)\exp\left[\left(a^\dagger a+b^\dagger b+1\right)\ln\left(\frac{1}{s''^*}\right)\right]\exp\left(-\frac{r''^*}{s''^*}ab\right), \end{aligned} \tag{4}$$

where

$$r''=sr'+rs'^*,\; s''=ss'+r'r'^*. \tag{5}$$

By observing (4), one can see that the unitary condition still holds for s'' and r''.

Now as the result of (2), we act $U_2(r,s)$ on a two-mode vacuum state $|00\rangle$ and obtain the explicit expression of GTSVS, i.e.,

$$U_2(r,s)\,|00\rangle=\frac{1}{s^*}\exp\left(\frac{r}{s^*}a^\dagger b^\dagger\right)|00\rangle\,. \tag{6}$$

In particular, when $s=\cosh\lambda$ and $r=\sinh\lambda$, Equation (6) becomes the ideal EPR entangled state with λ being the squeezing parameter,

$$U_2(\sinh\lambda,\cosh\lambda)\,|00\rangle=\operatorname{sech}\lambda\exp\left(\tanh\lambda a^\dagger b^\dagger\right)|00\rangle \tag{7}$$

which is just well-known two-mode squeezed vacuum state. Therefore, (6) is called the generalized two-mode squeezed vacuum state.

3 Entangling Properties of Output Field from BS for GTSVS Inputs

We know that the input field described by the operator a is superposed on the other input field with operator b by a lossless symmetric BS, which can be described by a unitary transformation operator [28]

$$B=\exp\left[\theta\left(a^\dagger be^{i\phi}-ab^\dagger e^{-i\phi}\right)\right], \tag{8}$$

with the amplitude reflection and transmission coefficients

$$r=\sin\theta,\quad t=\cos\theta \tag{9}$$

where ϕ is the phase difference between the reflected and transmitted fields. It is clear that

$$B\,|00\rangle=|00\rangle\,, \tag{10}$$

where $|00\rangle$ is the two-mode vacuum state. Equation (10) is due to the simple fact of no input no output. More generally, using the operator identity

$$\exp(C)\, D \exp(-C) = D + [C, D] + \frac{1}{2!}[C, [C, D]] + \frac{1}{3!}[C, [C, [C, D]]] + \cdots \tag{11}$$

the well-known transformation relations are displayed regarding the BS operator given by (8), i.e. [19]

$$B^{\dagger}\begin{pmatrix} a \\ b \end{pmatrix} B = \begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ -\sin\theta e^{-i\phi} & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}. \tag{12}$$

Next, we are interested in the entanglement properties of the output state. Suppose that ρ_{out} and ρ_{in} are density operators for input and output states, which reads as

$$\rho_{out} = B\rho_{in}B^{\dagger} \tag{13}$$

together with the fact that $B^{-1} = B^{\dagger}$ owing to the unitarity of the BS operator. When the input state is GTSVS, then the density operator can be written as

$$\rho_{in} = U_2(r, s)\,|00\rangle\langle 00|\,U_2^{\dagger}(r, s). \tag{14}$$

For simplicity, we use the characteristic function for the input state ρ_{in} and the output state ρ_{out}. For the GTSVS input, using (6) and (14) the corresponding characteristic function is given by

$$\begin{aligned}\chi_{in}(\alpha, \beta) &= \mathrm{Tr}\left[\rho_{in}D(\alpha, \beta)\right] \\ &= \exp\left[-\frac{1}{2}\left(\left|\alpha s^* - \beta^* r\right|^2 + \left|\beta s^* - \alpha^* r\right|^2\right)\right],\end{aligned} \tag{15}$$

where $D(\alpha, \beta) = \exp(\alpha a^{\dagger} - \alpha^* a + \beta b^{\dagger} - \beta^* b)$ denotes the displacement operator and we have also considered the relation

$$U_2^{\dagger} D(\alpha, \beta)\, U_2 = D\left(\alpha s^* - \beta^* r, \beta s^* - \alpha^* r\right). \tag{16}$$

By taking $\beta = 0$, the characteristic function of the input state in mode a can be expressed as

$$\chi_{in}^{(a)}(\alpha) \equiv \chi_{in}(\alpha, \beta = 0) = \exp\left[-\frac{1}{2}|\alpha|^2\left(1 + 2|r|^2\right)\right]. \tag{17}$$

In the case of the GTSVS inputs, the characteristic function for output state $\rho_{out} = B\rho_{in}B^{\dagger}$ is

$$\begin{aligned}\chi_{out}(\alpha, \beta) &= \mathrm{Tr}\left[\rho_{out}D(\alpha, \beta)\right] \\ &= \langle 00|\, D\left(\bar{\alpha}s^* - \bar{\beta}^* r, \bar{\beta}s^* - \bar{\alpha}^* r\right)|00\rangle \\ &= \exp\left[-\frac{1}{2}\left(\left|\bar{\alpha}s^* - \bar{\beta}^* r\right|^2 + \left|\bar{\beta}s^* - \bar{\alpha}^* r\right|^2\right)\right] \\ &= \exp\left\{-\frac{1}{2}\left[\left(|\bar{\alpha}|^2 + \left|\bar{\beta}\right|^2\right)|s|^2 + \left(|\bar{\alpha}|^2 + \left|\bar{\beta}\right|^2\right)|r|^2 - 2\bar{\alpha}^*\bar{\beta}^* rs - 2\bar{\alpha}\bar{\beta}r^*s^*\right]\right\},\end{aligned} \tag{18}$$

where we have set

$$\bar{\alpha} = \alpha \cos\theta - \beta \sin\theta e^{i\phi}, \qquad \bar{\beta} = \alpha \sin\theta e^{-i\phi} + \beta \cos\theta \tag{19}$$

and used (16).

Because continuous variable entanglement, generated by BSs experimentally, results in Gaussian state which can be transformed through an appropriate local linear unitary Bogoliubov operator (LLUBO) [29] to the standard forms given by (15) and (18). Thus it follows from (18) that the characteristic function for the output state in mode a from the BS is

$$\begin{aligned}
\chi_{out}^{(a)}(\alpha) &= \chi_{out}(\alpha, \beta = 0) \\
&= \exp\left[-\frac{1}{2}\left(\left|\bar{\alpha}s^* - \bar{\beta}^* r\right|^2 + \left|\bar{\beta}s^* - \bar{\alpha}^* r\right|^2\right)\right] \\
&= \exp\Big\{-\frac{1}{2}\left(\alpha_1^2 + \alpha_2^2\right)\left(|s|^2 + |r|^2\right) + (\alpha_1 - i\alpha_2)^2 \cos\theta \sin\theta (\cos\phi + i\sin\phi) rs \\
&\quad + (\alpha_1 + i\alpha_2)^2 \cos\theta \sin\theta (\cos\phi - i\sin\phi) r^* s^*\Big\} \\
&= \exp\left\{-\frac{1}{2}(\alpha_1, \alpha_2) M (\alpha_1, \alpha_2)^T\right\}
\end{aligned} \tag{20}$$

where $\alpha = \alpha_1 + i\alpha_2$, $s = s_1 + is_2$ and $r = r_1 + ir_2$, as well as M is the 2×2 correlation matrix as

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{21}$$

whose elements are, respectively,

$$\begin{aligned}
m_{11} &= (r_1 s_2 + r_2 s_1)[\cos(2\theta + \phi) - \cos(2\theta - \phi)] \\
&\quad + (r_1 s_1 - r_2 s_2)[\sin(2\theta + \phi) + \sin(2\theta - \phi)] - \left(s_1^2 + s_2^2 + r_1^2 + r_2^2\right),
\end{aligned} \tag{22}$$

$$\begin{aligned}
m_{22} &= (r_1 s_2 + r_2 s_1)[\cos(2\theta - \phi) - \cos(2\theta + \phi)] \\
&\quad + (r_2 s_2 - r_1 s_1)[\sin(2\theta + \phi) + \sin(2\theta - \phi)] - \left(s_1^2 + s_2^2 + r_1^2 + r_2^2\right),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
m_{12} = m_{21} &= (r_2 s_2 - r_1 s_1)[\cos(2\theta + \phi) - \cos(2\theta - \phi)] \\
&\quad + (r_1 s_2 + r_2 s_1)[\sin(2\theta + \phi) + \sin(2\theta - \phi)].
\end{aligned} \tag{24}$$

We know that any unitary transformation U acting on the single mode state ρ_a does not change the value of the entanglement entropy [30], i.e., $E(\rho' = U\rho U^\dagger) = E(\rho)$. Using this property, we can transform the output state ρ_{out} of BS via LLUBO into the entropy-equivalent state ρ'_{out}, whose characteristic function can be written as the standard form

$$\tilde{\chi}_a(\alpha) = \exp\left[-\frac{1}{2}(\alpha_1, \alpha_2)\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}(\alpha_1, \alpha_2)^T\right], \tag{25}$$

and

$$\lambda = \sqrt{\det M} = \sqrt{m_{11} m_{22} - m_{12}^2}. \tag{26}$$

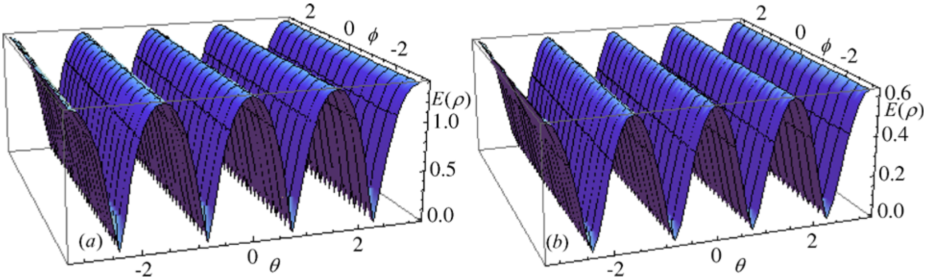

Fig. 1 The entanglement entropy of output state from BS as a function of the parameters θ and ϕ fixed at $r_2 = s_2 = 0$ for the different values (**a**) $r_1 = 1$ and (**b**) $r_1 = 0.5$

Clearly, the state defined by the characteristic function of (25) takes on a form of thermal state

$$\rho'_{out} = \left(1 - e^{-\eta}\right) e^{-\eta a^\dagger a}, \tag{27}$$

with its characteristic function given by

$$\chi_{th}(\alpha) = \exp\left[-\frac{1}{2}(\alpha_1, \alpha_2)\begin{pmatrix} \frac{1+e^{-\eta}}{1-e^{-\eta}} & 0 \\ 0 & \frac{1+e^{-\eta}}{1-e^{-\eta}} \end{pmatrix}(\alpha_1, \alpha_2)^T\right]. \tag{28}$$

Comparing (25) with (28), we find the parameter η satisfying

$$e^{-\eta} = \frac{\lambda - 1}{\lambda + 1}. \tag{29}$$

After some calculation, the entanglement entropy is given by

$$E\left(\rho_{out}^{(a)}\right) = -\mathrm{Tr}\left(\rho' \ln \rho'\right) = -\ln\frac{2}{\lambda+1} - \frac{\lambda-1}{2}\ln\frac{\lambda-1}{\lambda+1}, \tag{30}$$

where we have employed the fact that the trace values does not change under any unitary transformation. From (30) it is easily found that the entanglement entropy is completely determined by λ.

In order to observe the influence of entanglement entropy governed by its parameters, the numerical results for (30) are plotted in Figs. 1–3. From Fig. 1 we find that the degree of entanglement is periodic function for parameter θ. Then it seems that the smaller the squeezing parameter r, the larger the amplitude of degree of entanglement of the output state. Figure 2 shows that the degree of entanglement is an increasing function of the parameter r for the given argument θ determined by BS itself. It is seen from Figs. 1–3 that the degree of entanglement do not be impacted by angle parameter ϕ for the given other parameters.

4 Conclusions

In summary, with the help of the characteristic function, we have studied the entangling properties of output field from BS with the GTSVS inputs by introducing linear entropy as a measure of the entanglement. From the figures of the degree of entanglement, we find that

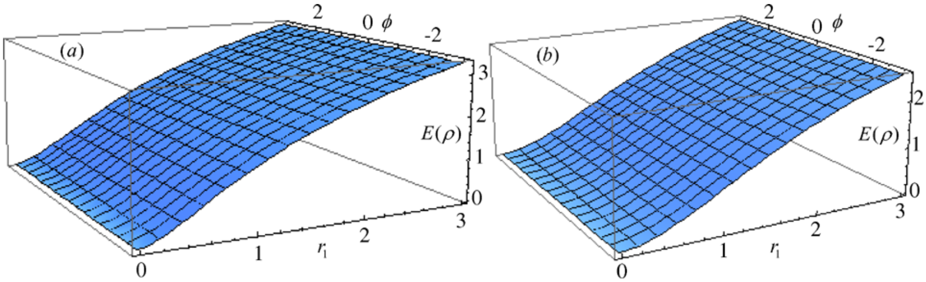

Fig. 2 The entropy of entanglement of output state from BS against the parameters r_1 and ϕ fixed at $r_2 = s_2 = 0$ for the different values (**a**) $\theta = 0$ and (**b**) $\theta = \pi/3$

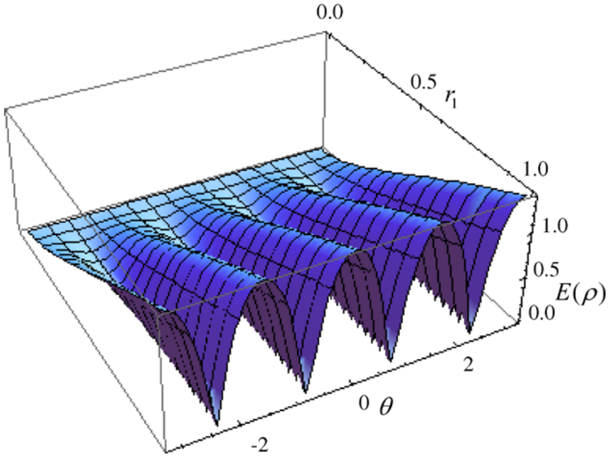

Fig. 3 The entropy of entanglement of output state from BS against the parameters r_1 and θ for the given value $\phi = 0$

degree of entanglement of the output state from the BS directly depends on the parameters of the input field and the BS. Based on the above discussions, our results provide a matched approach for manipulating entanglement by adjusting the parameters of input field and the BS, the best entanglement can be obtained when the parameters of both input states and the BS are chosen appropriately.

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References

1. Bennett, C.H., DiVincenzo, D.P.: Nature **404**, 247 (2000)
2. Bennett, C.H., Wiesner, S.J.: Phys. Rev. Lett. **69**, 2881 (1992)
3. Bennett, C.H., Brassard, G., Crepeau, C., Jozsa, R., Peres, A., Woottersand, W.K.: Phys. Rev. Lett. **70**, 1895 (1993)
4. Ekert, A.K.: Phys. Rev. Lett. **67**, 661 (1991)
5. Kimble, H.J., Walls, D.F.: J. Opt. Soc. Am. B **4**, 1449 (1987)
6. Reck, M., Zeilinger, A., Bernstein, H.J., Bertani, P.: Phys. Rev. Lett. **73**, 58 (1994)
7. Paris, M.G.A.: Phys. Rev. A **59**, 1615 (1999)
8. Zhou, L., Kuang, L.M.: Phys. Lett. A **302**, 273 (2002)

9. Mollow, B.R., Glauber, R.J.: Phys. Rev. **160**, 1076 (1967)
10. Schleich, W.P.: Quantum Optics in Phase Space. Wiley/VCH, New York (2001)
11. Tan, S.M., Walls, D.F., Collett, M.J.: Phys. Rev. Lett. **66**, 252 (1991)
12. Sanders, B.C., Lee, K.S., Kim, M.S.: Phys. Rev. A **52**, 735 (1995)
13. Kim, M.S., Son, W., Buzek, V., Knight, P.L.: Phys. Rev. A **65**, 032323 (2002)
14. Toth, G., Simon, C., Cirac, J.I.: Phys. Rev. A **68**, 062310 (2003)
15. Li, H.R., Li, F.L., Yang, Y.: Chin. Phys. **15**, 2947 (2006)
16. Marian, P., Marian, T.A., Scutaru, H.: J. Phys. A **34**, 6969 (2001)
17. Zhou, Q.P., Fang, M.F.: Chin. Phys. **13**, 1477 (2004)
18. Wang, X.B.: Phys. Phys. Rev. A **66**, 024303 (2002)
19. Wang, X.B.: Phys. Phys. Rev. A **66**, 064304 (2002)
20. Feng, X.L., Li, R.X., Xu, Z.Z.: Phys. Rev. A **71**, 032335 (2005)
21. Tahira, R., Ikram, M., Nha, H., Zubairy, M.S.: Phys. Rev. A **79**, 023816 (2009)
22. Fan, H.Y., Chen, J.H.: Commun. Theor. Phys. **38**, 147 (2002)
23. Fan, H.Y., Lu, H.L.: Phys. Lett. A **334**, 132 (2005)
24. Fan, H.Y.: J. Opt. B, Quantum Semiclass. Opt. **5**, R147 (2003)
25. Fan, H.Y., Lu, H.L., Fan, Y.: Ann. Phys. **321**, 480 (2006)
26. Fan, H.Y., Wang, T.T.: Int. J. Theor. Phys. **48**, 441 (2009)
27. Hu, L.Y., Fan, H.Y.: J. Mod. Opt. **55**, 2011 (2008)
28. Campos, R.A., Saleh, B.E.A., Teich, M.C.: Phys. Rev. A **40**, 1371 (1989)
29. Simon, R.: Phys. Rev. Lett. **84**, 2726 (2000)
30. Böhmer, B., Leonhardt, U.: Opt. Commun. **118**, 181 (1995)